Solving the Math Anxiety Problem Before It Starts

Patricia D. Stokes and Andrew Sanfratello

Barnard College, Columbia University and CUNY

"Can we do math instead of watching the movie?"

Kindergartener

Abstract: *A student did ask that question. A whole class enthusiastically did math instead of watching the movie (Cardinale, 2019). Where? In a public school in Lodi, New Jersey, where kindergarteners learned how to think like mathematicians: in numbers, symbols, and patterns (DiLeo & Stokes, 2019). The learning was based on an early math intervention using an explicit base-10 count, a single manipulative, and deliberate practice (Stokes, 2014a, 2014b, 2016a), and its expansion from one school (the pilot) to five (the district). The intervention was designed using a problem-solving model of creativity/innovation (Stokes, 2006, 2016b). This paper reports on an equally successful expansion in first grade. It is the first to report success in two ways: acquiring the math and not acquiring the anxiety. Problems with, and implications for, early math curricula are discussed.*

INTRODUCTION

Mathematical performance depends on working memory. Math anxiety – a negative emotional response associated with hyperactivity in the right amygdala (Young et al., 2012) – has been shown to impact mathematical performance by interfering with/compromising working memory (Ashcraft & Kirk, 2001; Beilock, 2008; Hembree, 1990; Vukovic et al., 2013) in tasks ranging from retrieval in elementary students (Ramirez et al., 2013) to computational span tasks in college students (Ashcraft & Krause, 2007). Given the strong relationship between a child's working memory span and mathematical performance (Adams & Hitch, 1997), math anxiety is presumably a factor in low early math scores. Given the converse – that low early math scores are a factor in math anxiety (Meece, Wigfield, & Eccles, 1990) – Harari et al. (2013) examined several dimensions of math anxiety in first graders. Their results showed that one dimension (negative feelings) was related to lack of mastery in "foundational mathematical concepts" such as counting, while another (numerical confidence) was related to lack of mastery in computational skills, like addition.

Counting and computation are basic skills, which makes it puzzling that there are (to our knowledge) no studies relating math anxiety to what we perceive as its root cause: early math curricula that do not foster mastery in numeric-symbolic patterns. What sorts of patterns? One example is the commutative property of addition: numbers can be added in any order to get the same answer. For example, $4 + 2 = 2 + 4 = 6$. The commutative property of multiplication is similar: numbers can be multiplied in any order to get the same answer. For example, $4 \times 2 = 2 \times 4 = 8$.

Mathematicians think in these kinds of patterns. The current intervention was designed to teach children how to think like mathematicians. Its name, *Only the NUMBERS Count©*, summarizes the strategy behind its success. The strategy is simple: *immersion in the strictly mathematical,* i.e., numbers, symbols, and patterns. Since the strategy was based, not on math education *per se*, but on expertise and problem solving, we begin with a brief discussion of expertise before describing the problem-solving model used to design the intervention.

EXPERTISE

Expertise is domain-specific. An expert is someone who has mastered the materials that define their area of expertise. Experts know more than novices, notice more than novices (Stokes & Gibbert, 2019), and construct more effective problem spaces than novices (Weisberg, 2006) because what they know is organized in ways that facilitate problem solving. That organization is based on patterns as well as on understanding the relationships that underlie those patterns.

Expertise and Patterns. Experts think and problem solve using large meaningful patterns in the *languages of their domains* (Chi, 2011; Chi, Glaser, & Farr, 1972). For diagnosticians, the patterns are represented by symptoms, behavioral and biological; for composers, by scales, pitches, rhythms, and sonorities; for chess masters, by legitimate board arrangements.[7] For mathematicians, the patterns are represented by numbers and symbols.

These numeric-symbolic patterns are stored in long-term memory in what Chase and Ericsson (1982) called "retrieval structures." The structures provide "slots" for rapidly storing relevant information in long-term memory. The structures are analogous to what are now called associative networks. The slots correspond to nodes connected to other nodes with related content. For example, a novice network for addition might contain the nodes and connections shown in Figure 1. The "flip" is what children (in our program) call the commutative property of addition, e.g., numbers can be added in any order. When they are more expert, the mathematically precise term would be added to their network.

[7] A legitimate arrangement is a board position reached by correct moves of each chess piece (Chase & Simon, 1973).

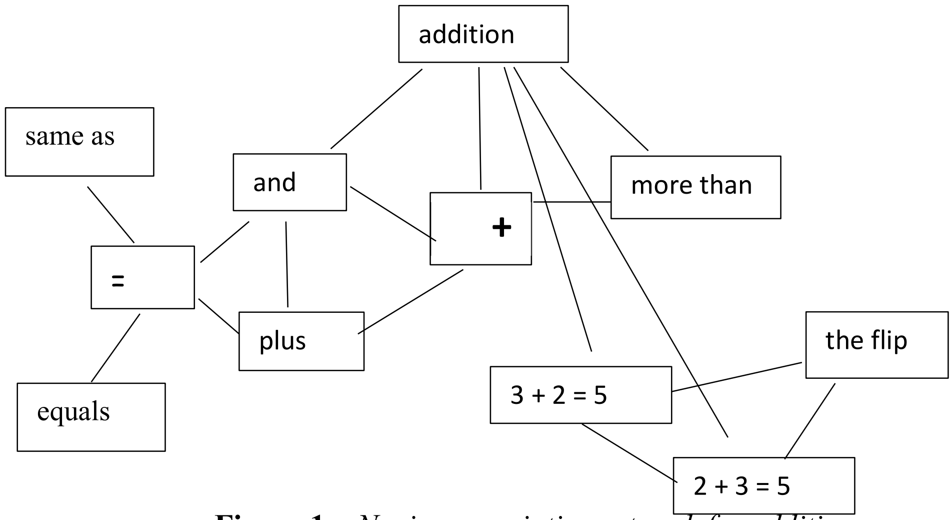

Figure 1. *Novice associative network for addition.*

Now, imagine the network expanding when our novice also learns that 2 + 3 also equals 2 + 2 + 1 which equals 4 + 1 or 1 + 4, all of which equal 5. (Notice how, in this way, procedural knowledge generates conceptual knowledge: numbers are combinations of other numbers.) Imagine it expanding further when our novice learns that subtraction "un-does" addition. To "un-do" is analogous to the more mathematically precise process of finding the "inverse." In each case, the result is taking the original output value and mapping the solution back to its original input value. An expert would have "inverse" (and its connections to addition/subtraction and multiplication/division) in their associative network.

Compared to our novice's, an expert mathematician's network would be extensively patterned and integrated, allowing the expert to readily access and retrieve those patterns[8] (Ericsson & Kintsch, 1995), and to efficiently expand and elaborate them (Nokes, Schunn, & Chi, 2019). The more you know, the easier it is to know more.

Expertise and Deliberate Practice. In well-established domains, the patterns that constitute expert knowledge are acquired (in a semi-established order) in a process called deliberate practice (Ericsson, 2006). Deliberate practice is focused (on specific patterns), continuous (in developing those patterns) and variable (in elaborating them). Deliberate practice is how procedural knowledge is acquired and expanded.

[8] The ability to overcome the capacity limits of short-term memory is called skilled memory or Long-Term Working Memory (Ericsson & Kitsch, 1995).

THE PROBLEM-SOLVING MODEL

The model is based on what Newell and Simon (1972) called a problem space. A problem space has three parts: an *initial state*, a *goal state*, and, between the two, a *search space*, in which *paired constraints* structure a solution path that changes the initial into the goal state (Reitman, 1965; Simon, 1973; Stokes, 2006). One of the paired constraints precludes something specific in the initial state; the other promotes a substitute. The process is called *solution-by-substitution*. The process is how the curriculum was created.

As shown in Table 1, the initial state was current early math curricula, characterized by elements in the preclude column of the search space. The goal state was a new curriculum with a very specific criterion, teaching children to think in numbers, symbols, and patterns. The new curriculum is characterized by the substitutions shown in the promote column. The promote column is the solution path. We consider each preclude-promote pairing in turn.

Table 1. Problem-space for *Only the NUMBERS Count©*

Initial State:	Current early math curricula		
Search Space:	Paired constraints		
	Preclude	→	***Promote***
	Non-numeric		Primacy of numbers, symbols, and patterns
	Current count		Explicit base-10 count
	Multiple manipulatives		Single manipulative
	Split practice		Deliberate practice
Goal State:	New curriculum		
Criterion:	Thinking in numbers, symbols, and patterns		

Primacy of Numbers, Symbols, and Patterns. There are two reasons to preclude the non-numeric from early math education. First, we want children to begin thinking like mathematicians. Experts problem-solve in the language of their domain (Chi, 2011; Chi et al., 1986). For mathematicians, the language is numbers, symbols, and patterns, not words. Fluency in a language depends on time of introduction – early is important – and time spent practicing –

immersion is important (Johnson & Swain, 1997). Second, to solve a word problem, you need a mathematical model on which to map it. This is why we wait until later in the school year to teach children how to 'translate' word problems into math problems.

Explicit Base-10 Count. English number names were precluded because they obscure the base-10 pattern of the count. **In their place, we substituted number names based on Asian language counts, which make the patterning explicit. Our** English language version of the explicit base-10 Asian count is shown (in part) in Table 2. There are four things we want to emphasize. One, the first ten number names (1 to 10) combine in ***iterative patterns*** to form higher numbers. Two, this makes every number name ***quantitatively concrete***. Three, ten appears in every number above ten up to 100 (e.g., 12 is ten-two, 22 is two-ten two). Four, ten is a unit. It is not ten ones, it is one ten.[9]

Table 2. *Explicit base-10 count in English.*

Ones	Tens	Twenties	Etc.
	10 *ten*	20 two-*ten*	30 three-*ten*
1 one	11 *ten*-one	21 two-*ten*-one	31 three-*ten*-one
2 two	12 *ten*-two	22 two-*ten*-two	32 three-*ten*-two
3 three	13 *ten*-three	23 two-*ten*-three	……..
4 four	14 *ten*-four	24 two-*ten*-four	44 four-*ten*-four
5 five	15 *ten*-five	25 two-*ten*-five	45 four-ten-five
6 six	16 *ten*-six	26 two-*ten*-six	……..
7 seven	17 *ten*-seven	27 two-*ten*-seven	57 five-*ten*-seven
8 eight	18 *ten*-eight	28 two-*ten*-eight	58 five-*ten*-eight
9 nine	19 *ten*-nine	29 two-*ten*-nine	…….

The Single Manipulative. We precluded multiple manipulatives because they are distractions: the things counted (10 *straws*, 10 *paper clips*) are more salient than the commonality

[9] As a consequence of their count, Asian children think of numbers as combinations of 10s and 1s. This eliminates the place-value problem (Fuson, 1990).

of the count (*10*). Our substitute, called the Count-and-Combine Chart, remedies this problem by borrowing two things from the abacus: first, it makes the base-10 patterns visible, tangible, and ***concrete***; second, it ***only*** represents numbers and patterns.

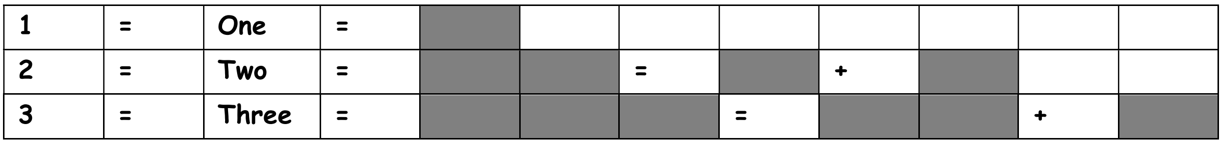

Figure 2. *Three lines from the 1 to 10 Count-and-Combine Chart.*

Figure 2 shows the first three lines of the 1 to 10 Count-and-Combine Chart, all parts of which were moveable. Children recited the rows this way: "Number 1 same as word one equals one block. Number 2 same as word two equals two blocks…"[10] They arranged and re-arranged the blocks (laminated poster board) to make combinations, like the ones shown on lines 2 (1 + 1) and 3 (2 + 1). They also used bags of loose "blocks" to make *addition combinations* at their tables to supplement their work with the Count-and-Combine Chart. The word 'combination' was used to emphasize the fact that numbers are combinations of other numbers.

Why is our count so complicated?

After the first decade of numbers (from 1 through 9), it would make the most sense for the numbers 11 through 19 to linguistically mimic the numbers 1 through 9. This would make the obvious connection to emphasize our base-10 system. Unfortunately, due to the variances and evolution of language, the connection is not obvious at all.

The words eleven and twelve trace their etymologies back to the terms "one left" (over ten) and "two left" (over ten), respectively (and also reflect the remaining traces of a base-12 number system no longer utilized today). The words thirteen through nineteen are more or less bastardizations of three-ten, four-ten, five-ten, etc., which have evolved over generations. For the decades that follow there does exist a stronger connection to the first decade of numbers. Each two-digit number from 21 through 99 requires only the knowledge of the word for the tens digit (e.g., twenty, thirty, forty, etc.) followed by the single digit name word for the units digit. Each word for the tens digit needs to be committed to memory for the learner, which each bear some commonalities of their stem (e.g., two and twenty; three and thirty; four and forty; etc.), but once these are known, the pattern repeats itself each decade. Beyond two-digit numbers, the language gets more streamlined. We do not have a special word for, say, 600 or 6,000, the way we do for 60, merely the name for the single digit, followed by the place value.

[10] Notice that the equals sign is alternatively called 'same as' and 'equals.' The first term defines the second.

This obfuscation is not unique to the English language either. Romance languages, American Sign Language, Farsi, and Hindi each behave similarly to English, in that for numbers below 20 there is complexity, while beyond the number 20, there is uniformity; though French has even more peculiarities in number names above 70. In German, the ambiguity lies between the numbers 21 and 99, where the numbers are read more or less from the units digit to the tens digit, so the number 54 is roughly four and fifty.

Fortunately, children learn language easily and, as shown in our classrooms, quickly become fluent in the explicit base-10 count.

Figure 3 shows the lines for 11 (ten-one), 12 (ten-two), and 13 (ten-three) in the Ten to Two-Ten (20) chart. Notice that ten is represented by a single block marked '10.' One combination appears on each line. For 13, the combination shown is ten-one plus two. As the school year progressed, children were also able to decompose the ten, for example, 6 + 4 + 3 = 13. Once children mastered the combinations for 1 through 10, *subtraction* was taught[11] conceptually (as un-doing addition) and procedurally (using green blocks for the minuend and red ones for the subtrahend). Children repeated the phrase "subtraction means taking away" as they simultaneously removed one red block and one green block until only green blocks were "left." When solving problems with 10s and 1s, they learned to "take away" the 10 blocks first.

10	=	Ten-one	=	10		=	10	+					
12	=	Ten-two	=	10			=	10	+				
13	=	Ten-three	=	10				=	10		+		

Figure 3. *Three lines from the Ten to Two-Ten Count-and-Combine Chart.*

Deliberate Practice. The precluded practice is aptly named. Split (intermittent) practice switches between kinds of problems (a little addition, a short introduction to subtraction, a little more addition…) and materials (those multiple manipulatives), making practice on any skill partial and interrupted. In contrast deliberate practice is highly focused on specific aspects of a skill to be continuously developed in highly variable ways (Ericsson, 2006). Decomposing (or to be more mathematically precise, partitioning) numbers provides a good example of deliberate practice.

As the numbers increase, so do the number of possible decompositions. The number 3 has four decompositions (3, 2 + 1, its flip 1 + 2, and 1 + 1 + 1). Students learn to associate 2 + 1 and 1 + 2 as "flips" of one another. We prefer this more colloquial term for the mathematical property of commutativity here.

The number 4 has eight decompositions (4, 3 + 1 and its flip, 1 + 3, and also 2 + 2, the latter of which can be further decomposed into 2 + 1 + 1, or 1 + 1 + 2, or 1 + 2 + 1, all of which can be decomposed into 1 + 1 + 1 + 1).

The number 5, has sixteen decompositions, and this, along with a more general formula, is seen below. Because the children learn all these decompositions (by physically constructing

[11] Children in two of the four expansion schools had already done subtraction (this way) in kindergarten.

them[12]) early in the school year, and because each decomposition builds on previous decompositions of smaller numbers, none of the combinations in the previous paragraphs would be a problem – or a cause of anxiety.

Deliberate practice is how immersion happens.

Decomposing = Partitioning

The process of decomposing positive integers that students are practicing in this curriculum is known as "partitioning" in Number Theory. Partitioning has deep connections to a variety of areas of mathematics. In general, the number of partitions of a number, n, is equal to 2^{n-1}. For example: The number $n = 5$ has $2^{5-1} = 2^4 = 16$ different partitions. These 16 partitions can be organized, like in the table below, by the amount of integers in the partition. Astute mathematicians will recognize the connections that exist between these partitions and the Binomial Theorem, and in fact, a bijection exists.

Addends in the partition	Partition	Number of possible partitions
1	5	**1**
2	$4+1$ $1+4$ $3+2$ $2+3$	**4**
3	$1+1+3$ $1+3+1$ $3+1+1$ $2+2+1$ $2+1+2$ $1+2+1$	**6**
4	$1+1+1+2$ $1+1+2+1$ $1+2+1+1$ $2+1+1+1$	**4**
5	$1+1+1+1+1$	**1**

[12] Learning combinations by making them is an example of Simon's (1988) "learning by doing."

THE PRESENT STUDY

We report on the expansion of *Only the NUMBERS Count©* to all first-grade classes in the Lodi, NJ school district. The first set of results covers classroom observations and teacher surveys. The second presents pre- and post-test results, along with examples and explanations, from two first grade classes in two different schools. The third presents first graders responses to our simple, direct math anxiety questions.

OBSERVATIONS AND TEACHER SURVEYS

Method

Participants. Participants in the expansion included four schools with 12 first grade classes, and 161 students. Three of these classes were designated special education: there were 10 students in these classes. First grade teachers in the fifth school, where the program was developed, had used it for several years. The fifth school had 4 first grade classes with 61 students. One class with 3 students was special educations. The special education classes were not included in the observations. Only two special education teachers completed the teacher survey.

Procedure. There was a professional development day at the start of the school year. Teachers new to the program met with the PI and experienced teachers at one of the grammar schools. The new teachers were given how-to-work books with lesson plans, made their own materials (Count-and-Combine charts, poster-board 'blocks' for children to use) and learned how to use them. Instruction included how to talk about math using the explicit base-10 count.

School Visits. The number of visits was limited due to the number of schools, as well as days off and breaks (for both the district and the university). Visits did not begin until early October. By the end of the school year, all schools had been visited 4 times. Pre- and post-testing were done at the start and end of the school year by the PI and assistants. Teacher and student surveys were also done close to the end of the year. The teacher surveys were done online; the student surveys were conducted in the classrooms by the PI or the teachers.

Results

Observations. All classes began reviewing the 1-to-10 chart and making combinations with the blocks representing ones. Table 3 shows when more advanced tasks (using the 10-to-20 chart, breaking up/decomposing combinations, subtraction) were introduced in the four schools. *Decomposing* meant taking a combination that a child made (say, 3 + 4 = 7) and breaking up either the 3 or the 4 to make the combination "longer" (3 + 2 + 2 = 5). Children could then make a new "shorter" combination" by recombining the numbers (5 + 2 = 7). When subtracting, children physically "took away" the same number of blocks from either side of the minus sign. Near the end of the school year, higher performing students worked on double-digit addition and subtraction, while lower performing students worked on single-digit problems.

Table 3. *Task introduction by school.*

Task	*January/February*	*February/March*	*April/May*
10-to-20 chart	Roosevelt/Hilltop Wilson	Columbus	
Decomposing combos			
Making "longer"	Roosevelt/Hilltop	Wilson	
Making "shorter"	Roosevelt/Hilltop		
Subtraction			
With 10s and 1s	Roosevelt/Hilltop	Wilson	Columbus

The differences in the chart appear to be teacher specific. Roosevelt School had the advantage of having two teachers who had used *Only the NUMBERS Count©* in kindergarten and were familiar with the program. Teachers at Hilltop and Wilson enthusiastically embraced the program from the start. However, one teacher at Wilson only began substituting (and thus using the program) in the spring. One teacher at Columbus began quite slowly, the other didn't begin until later in the school year.

At the end of the school year, we visited two first grade classes in Washington School where the program was piloted and, hence, well established. Both teachers said they had completed the first-grade curriculum and were preparing their classes for 2nd grade. One class had begun using the multi-operation chart developed for 2nd grade. The chart is designed to teach multiplication and division simultaneously.

Teacher Survey. All teachers in the non-special education math classes filled out the survey, which is shown in the Appendix. The most important results can be collapsed into two questions:

1. Do you like teaching *Only the NUMBERS Count©?*

a. 12 teachers answered yes; only 1 answered no. This means that ***92% liked the program**,* only ***8% didn't.*** Liking suggests that teachers were not anxious about using the program.

2. What do you like about it?
   a. Most frequent answers were variations of:
      - The hands-on learning.
        - Students were excited/engaged.
        - "Doing things hands-on gives them great confidence."
      - Students understood the meaning of what they were learning.
        - "They could visually see what they were solving."
        - "They appear to think about their thinking."
        - *"They were thinking like mathematicians."*
        - "It helps them better visualize math."
        - "Students were able to grasp the concepts it teaches."
      - Students understood place-value.

   b. One teacher liked that "It is built upon. They had a foundation from last year."

PRE- AND POST-TESTING THE CHILDREN

Since we only visited one school per week, there was insufficient time to pre- and post-test all children. Two classes in two different schools were selected randomly for this purpose.

Method

Participants. Twenty-seven children in two classes at two different schools in the district served as participants. Children were sorted into classes by ability. Both classes followed the New Jersey Math Standards. Both used material from *Only the NUMBERS Count*© for numbers and numeric relations, and materials from *enVisionMATH* for all other required topics. The time for math was equal in both groups. Descriptions of each group and its teacher follow.

Both teachers attended a preparatory workshop with the experimenter and teachers who had worked previously with the program. They made the materials needed (Count-and-Combine charts, etc.) at the workshop. To ensure fidelity of treatment, the PI and two to three assistants observed math lessons on a rotating basis. While the core elements of the program were preplanned, timing of the implementation depended on the teacher's assessment of when their children were ready to move on to more advanced materials.

Class 1. At the beginning of the school year, there were 11 students (6 female, 5 male) in the class. Of these, 4 were Hispanic or Latino, 6 were White, and 1 was Black or African-American. One was classified as ESL; none as economically disadvantaged. Mean age at the beginning of the year was 78.8 months; range was 72 to 83 months. The teacher was experienced using *enVisionMATH*. She had 24 years total experience. In this school district, she taught fifth grade for 1 year, fourth for 11 years, kindergarten for one year, and first grade starting with this class.

Class 2. At the start of the school year, there were 16 students (11 female, 5 male) in the class. Of these, 10 were Hispanic or Latino, 5 were White, and 1 was Asian. Eight were classified as economically disadvantaged (eligible for reduced fee lunch); 1 as ESL. Mean age at the start of the year was 80.9 months; range was 72 to 95 months. The teacher also had experience using *enVisionMATH.* She had 12 years teaching experience. In this district, she taught second grade for 1 year, and first grade starting with this class.

Procedure. The study had three phases. Phase 1 included pre-testing to assess what students retained from kindergarten. Phase 2 included class visits. Phase 3 included post-testing to assess what had been learned. Testing was done by the primary experimenter and undergraduate research assistants.

Phase 1. Assessing Prior Knowledge. Pre-testing took place on October 12th and October 18th 2018. Table 2 presents items on the pre-test. The test was identical to that given at the end of kindergarten to two different classes that were also exposed to *Only the NUMBERS Count©* in kindergarten. This was used to see how much children retained over the summer.

Phase 2. Observing the Classes. Both were visited four times between October and May.

Phase 3. Assessing New Knowledge. Table 3 presents items on the post-test. Notice that there were more and more difficult items than on the pre-test.

Table 4. *Pre-test items.*

Category	***Content***
Counting.	Children were asked to count as high as they could. Counting was coded as correct up to the first error (If a child counted 11, 12, 15, her score was 12, the highest correct number).

Number and symbol identification.	
	Children were asked to read aloud 10 written numbers (1, 2, 4, 5, 7, 8, 12, 15, 20, 32) and three symbols (plus, minus, equals) presented in problem format (e.g., 2 + 2 = 4).
	Correct responses for the + sign were: plus, and N more, add. Correct responses for the = sign were equals or same as. Correct responses for the – sign were: minus, less, and take away.
Place-value.	Children were asked (a) to read aloud the written numbers 16, 25, 31, 56, 11; (b) tell the experimenter which of each pair was bigger and (c) explain their answers.
Addition.	Children solved two single digit (3 + 5, 6 + 2) and two double digit (12 + 4, 21 +11) problems. They were asked (a) to read the problem, (b) solve it, and (c) tell the experimenter how they did it.
Subtraction.	Children solved two single digit (5-3, 7-5), and three double digit (10 – 6, 10 – 10, 22 - 12) problems. They were asked (a) to read, (b) solve and then (c) explain their method to the experimenter.
Combinations.	Children read two numbers (8, 12) aloud and were asked to make up addition problems (e.g., ___+___= 8) and explain how they did each one.

Table 5. *Post-Test Items*

Category	***Content***
Counting.	
To 100	Identical to pre-test.
By tens	Children were asked to count by 10 to 100.
How many tens.	
	Children were asked how many tens there were in 30 and in 50.
Number and symbol identification.	
	Identical to pre-test.
Place-value.	

Version 1 Identical to pre-test.

Version 2 Five different numbers (18, 27, 31, 58, 22) were used.

Children were asked (a) what is this number, (b) does it have any tens, (c) how many tens, (d) how many ones, and finally (e) which digit is bigger.

Addition. Children solved three single digit (3 + 5, 6 + 6, 9 + 7) and three double digit (10 + 18, 21 +11, 17 + 35) problems. They were asked (a) to read the problem, (b) solve it, and (c) tell the experimenter how they did it.

Subtraction. Children solved two problems with at least one single digit (5 - 3, 10 – 5, 13 - 6), and two double digit (20 - 20, 22 – 12) problems. They were asked (a) to read, (b) solve and then (c) explain their method to the experimenter.

Combinations. Children read three numbers (8, 10, 25) aloud and were asked to make up addition problems, two with 2 addends and one with 3 addends, and to explain how they came up, with each one. The three-addend problem could be solved by decomposing/partitioning one of the addends in the previous problem

Word Problems. Children read and figured out the answers to three addition problems and two two-step problems requiring (first) addition and (then) subtraction.

RESULTS

Given that the study involved a small number of students, we present descriptive statistics. We also present specific student solutions and explanations showing that their learning was primarily procedural.

How They Did: Pre- and Post-Test Scores.

Table 4 shows pre- and post-test scores for the two classes. We look first at pre-test scores.

Pre-Test Scores. Pre-test scores show that the children retained a great deal from kindergarten. The surprising exception was the relatively low place-value score in both classes. Relatively refers to post-testing at the end of kindergarten in two different classes. Means were 93.28 and 66.66. We attributed this decline to the time of testing. At the end of kindergarten, children would be reciting the 10 to 20 Count and Combine Chart (with 11 as ten-one) each day. At the start of first grade, they would not yet have reviewed this chart.

Table 6. *Pre- and Post-Test Scores.*

	Class 1				Class 2			
	Pre		Post		Pre		Post	
Measure	*Mean*	*SD*	*Mean*	*SD*	*Mean*	*SD*	*Mean*	*SD*
Count to 100 by 10s	85.63	*32.20*	97.27 100.00	*9.04* *.00*	81.80	*30.35*	95.87 95.00	*16.5* *20.00*
Tens in 30 & 50			95.45	*15.07*			68.75	*47.87*
Number-Symbol	98.36	*3.88*	100.00	*.00*	99.60	*1.54*	97.93	*6.433*
Place Value *Bigger* *How many 10s*	49.09	*45.04*	32.72 92.73	*46.70* *24.12*	28.00	*42.62*	32.50 85.00	*47.25* *30.55*
Addition Single Double	95.45 54.54	*15.07* *41.56*	100.00 60.45	*.00* *44.27*	93.33 43.44	*25.81* *31.99*	93.62 35.31	*13.70* *42.95*
Subtraction Single Double	77.27 54.09	*41.00* *30.65*	93.81 68.18	*13.75* *25.22*	96.66 68.26	*12.90* *8.77*	78.75 59.37	17.00 32.75
Combinations Doubles First Second Triples	90.90	*20.22*	93.81 90.81 66.54	*13.75* *21.72* *42.21*	63.33	*39.94*	74.62 64.31 39.43	*22.98* *33.35* *44.17*
Word Problems Addition Two step (add and subtract)			69.36 54.54	*27.86* *52.22*			70.43 40.62	*24.12* *41.70*

Post-test scores. We looked at each group alone to see where improvements occurred. We also compared performance between the two groups.

Class 1. This class was tested first. Scores increased in all categories except place value as tested using the "which digit is bigger" question. This could not be attributed to not being familiar with the explicit base-10 count. Since the children knew how many tens there were in 30 and 50, and had extensive practice using ten blocks and one blocks to add, subtract, and create combinations (all items on which they scored well), we needed to explain the anomaly.

To see if the problem was in the way the question was asked (which digit is bigger), we rephrased the question and re-tested this class at a later date. The re-phrasing, which reflected the class emphasis on breaking numbers into ten blocks and one blocks took this form:

What is this number?

Does it have any tens?

How many tens?

How many ones?

Which is the ten?

With the re-phrasing, place value scores rose to 92.72%: 10 of the 11 children scored 100%, one scored 20%.

Class 2. The new place value questions were tested at the same time as the other items in this group. Three scores increased: counting to 100 correctly, making one addition combination, and place value with the re-phrased question. Scores decreased in double-digit addition, and both single- and double-digit subtraction

Between Classes. Class 1 did noticeably better on: number of 10s in 30 and 50, double-digit addition, single- and double-digit subtraction, two-step word problems (with addition and subtraction), combinations (doubles and triples), and place value (with the re-phrasing). We attribute this difference to teacher enthusiasm and utilization of the program. The teacher in Class 2 resisted and rarely used the program.[13] Her class was eventually exposed to it by sitting in on math lessons in a class where the teacher was actively participating.

How They Did It: Explanations and Examples

Explanations. Since all testing was done one-on-one, we were able to ask each child how they solved each problem. Their explanations can be collapsed into two categories, shown by class, in Table 7. Counting took two forms: with the fingers or by drawing blocks. Notice that the class with the higher post-test scores (Class 1) had a higher percentage of students who said they "knew."

Table 7. *Explanations*

	Class 1	***Class 2***
I knew	**73%**	33%
Counted	27%	**66%**

Examples. What they knew was both practiced and procedural. Practiced refers to retrieval, e.g., "knowing" and recalling the addition face that 9 + 7 = 16. Procedural refers to knowing and performing the steps to find the sum. For some students, this meant mentally manipulating the ten and the one blocks, "knowing" to add or subtract the tens first, and then add or subtract the ones.[14] For others, it made sense to them to make multiple combinations with the

[13] She was the sole teacher who (on the survey) wrote that she did not like teaching *Only the NUMBERS Count©.*

[14] This procedure eliminates "carrying." For example, with 23 (two-ten-three) and 19 (one-ten-nine), adding the tens yields three-ten, adding the ones gets get ten-two. Three-ten plus ten-two is four-ten-two (42).

blocks, "knowing" to use a "double" (5 + 5 = 10) for a first combination, or to "flip" a first combination (5 + 3 = 8) to make a second (3 + 5 = 8).

Most impressive to us were the performances from students who were able to product a third combination. This ability to apply previous knowledge and to synthesize concepts indicated to us a deeper understanding of what it means to deconstruct numbers. Most students who showed this ability were in Class1. One student decomposed one number in their second combination, turning 2 + 8 =10 into 2 + 2 + 6 = 10. In the words of this student, "I broke apart the numbers and put in little numbers." In another example, explaining how 4 + 6 = 10 became 6 + 2 + 2 = 10, the student told us "I knew that 2 + 2 equaled 4."

The students who did well did so because they were fluent in numbers and symbols and patterns. They had learned to think like mathematicians.

MATH ANXIETY: ASKING THE CHILDREN

Math anxiety was not the original focus of this study. It was included at the end of the school year because virtually none was observed in our school visits. Rather, teachers (with the one exception) and children seemed to be engaged in, and enjoying, math. To quantify our observations, we reduced the MARS-E[15] (Suinn et al., 1988) to a single question: how do the children feel about math? We report data collected from 11 classes in the four expansion schools, and from 2 classes in the pilot school.

METHOD

Asking the Children. Due to scheduling of visits near the end of the school year, we could only ask children in 8 classes to choose one of three cartoon faces (smiley, neutral, sad) to show how they felt about math. To cover all the classes, the teachers asked their classes (all, including the 8 surveyed by the experimenters) two questions: How many of you like math? How many of you don't like math? Children who chose from the cartoon faces gave the same responses (smiley face = like, sad face = dislike) to their teachers' queries, making their ratings reliable.

But what about math anxiety? Of the 200 first grade children surveyed, 171 liked math, only 29 didn't. In percentages, this means that ***86% liked math***, only ***14% disliked it***.[16] These results would, to researchers studying math anxiety (e.g., Ramirez et al, 2013), be very surprising.

We asked the children who choose from the cartoon faces why they picked the smiley face. Their reasons included:

[15] Mathematics anxiety rating scale for elementary school students.
[16] We were also able to ask three kindergarten classes (also using *Only the NUMBERS Count*) how they felt about math. Of 54 students, 46 liked math, only 8 didn't. Put as percentages, ***85% liked math, 15% didn't,*** the results parallel those seen in 1st grade.

"The blocks make math easy."

"It's fun to learn."
"I like plus-ing and minus-ing."
"I like learning new things."

The answer we liked best was "*I like getting smarter*."

We also asked why the sad face was selected? One child simply said he didn't like math facts. Another said the math was too easy. The others didn't give any reasons.

Asking the Teachers. As already noted in the teacher survey section, ***92% liked the program, only 8% didn't***. This suggests that the teachers were not anxious about implementing and using the program.

DISCUSSION

Acquiring the Math. Observations, tests, and children's explanations show that the children did learn how to think in numbers, symbols, and patterns. This kind of thinking helped them meet all 1st grade standards for Operations and Algebraic Thinking (1.OA), and for Numbers and Operations in Base Ten (1.NBT). It also helped them learn things about numbers that they are not expected to learn (creating, decomposing, and recomposing combinations), or not expected to learn until 2nd grade – meeting all 2nd grade standards for Operations and Algebraic thinking (2.OA), and partially meeting those for Numbers and Operations in Base Ten (2.NBT.1 and 2.NBT.9)

Not Acquiring the Anxiety. Our survey asked if teachers liked teaching *Only the NUMBERS Count©.* One of the twelve teachers answered negatively. This meant that 92% liked the program, only 8% didn't. One reason for the low math anxiety in the students may be low anxiety in their teachers.

With the children, we operationalized math anxiety as choosing the sad face when asked how a child felt about math, or when they told their teachers that they liked or didn't like math. Only 14% of the children did not like math; 86% liked math. We underline this because it is markedly different from two recently reported studies with elementary students.

One study used an 8-item questionnaire and cartoon faces (calm, semi-nervous, nervous) to indicate (on a 16-point sliding scale) how anxious students felt about math. The mean math anxiety score was 8.07 (Ramirez et al, 2013). There were no 0s. All children experienced math anxiety. Another (Vukovic et al., 2013) did not report 2nd and 3rd grade scores on their math anxiety scale, but rather showed that those scores were negatively correlated to calculation and mathematical – but not to geometric – applications. The researchers then asked: why the difference? This is their answer. "Calculation skills and mathematical applications have in common that they are both based in the symbolic number system… [suggesting] that math anxiety may specifically affect mathematical problems that involve understanding and manipulating numbers" (p. 8). Perhaps, then, the reason we report so little math anxiety is immersion in manipulating numbers. We elaborate on this idea in the next section.

REASONS FOR THE RESULTS

The goal of *Only the NUMBERS Count©* is to have children problem-solve like mathematicians, thinking *not about* numbers, *but in* numbers, symbols, patterns. To do this requires ***immersion**,* that is, ***constant practice in manipulating numbers in order to understand them.*** Understanding, taken here in its problem-solving sense, is primarily procedural (Zhu & Simon, 1987).[17] It is also circular. Understanding numbers means knowing when and how to manipulate them.[18] To understand how the understanding happened – and the anxiety didn't – we look at the contributions of each of our core components.

The Explicit Base-10 Count. A basic problem with current curricula is not the often-blamed abstractness of math (Ashcraft & Krause, 2007), but the way the curricula *obscure* or *ignore* the concreteness and the connections. In contrast, using an explicit base-10 count teaches the children that numbers are specific real things that don't change, which makes them more stable and more substantial than the things to which they are *temporarily* attached. In other words, ***nothing is more concrete than the count.*** As an unpublished poem[19] put it:

You can count two snails or two pails,
two trucks or two ducks
Two can be one more than one (1 + 1 = 2)
or one less than three (3 – 1 = 2)
and also one half of four (4 ÷ 2 = 2)
But no matter what you do,
Two is always two
…and never more.

[17] In this view, conceptual knowledge emerges from procedural knowledge. For example, practice making and re-making 'combinations' with the blocks taught children how to (procedural) do addition, and also that (conceptual) numbers are combinations of other numbers.

[18] This definition of procedural knowledge is akin to the National Research Council's (2001) version: "knowledge of procedures, knowledge of when and how to use them appropriately" (p. 12).

[19] The poem goes through the count from 1 through 10. It was written by the first author.

Equally concrete are the count's ***patterned iterations*** that facilitate mastery of place-value and double-digit calculation.

The Count-and-Combine Charts and the Blocks. The charts were designed – like the abacus – to make numeric-symbolic patterns primary as well as concrete. Notice in the poem, there are three different ways to get to 2. The different ways illustrate a foundational pattern in mathematics: numbers are combinations of other numbers. The blocks were designed to let the children concretely (visibly, tangibly) practice combining pairs of numbers like 3 + 4 and 5 + 2, both of which generate the number 7. They soon practiced a related pattern, which generates two more combinations for 7: 4 + 3 and 2 + 5. Mathematicians call this pattern (order doesn't matter) the commutative property of addition. The children – thinking visually – call it the "flip." Importantly, the 10-block – which represents 10 as a unit, rather than as a grouping of ten ones – externalized place-value.

Deliberate Practice. Deliberate practice is how immersion happens. It is continuous, focused, and variable. The focus is on base-10 patterns and relations. The variability is the result of sustained, incremental elaboration of those patterns. The children practiced the pattern of the base-10 count by reciting the charts. They elaborated on the pattern by creating, decomposing, and recomposing addition combinations using the 10s and 1s of the count. They practiced reversing the pattern, using subtraction to un-do addition.

This kind of practice exemplifies the idea of learning by doing (Papert, 1980) or, more specifically, learning by solving problems (Zhu & Simon, 1987). In this view, the problem solving process itself provided "a template" on which "knowledge of a correct solution provides information not only about the steps that have to be taken to follow the path but also about the cues present in the successive situations reached that indicate which next steps may be appropriate" (Anzai & Simon, 1979, p. 137).

Immersion in the strictly mathematical means continuous practice in manipulating numbers in order to understand them. Practice using the blocks to make combinations (addition) and to physically "take away" (subtraction) certainly contributed to the OA results; practice using the Count-and-Combine charts and the explicit base-10 count, to the NBT results. Deliberate practice made the (successively more elaborate) procedures procedural.

Expertise is procedural. Expertise solves the math anxiety problem *before* it starts. If you can do the math, there is no reason to be anxious about doing it.

CAVEATS AND CONCLUSION

Caveats. Unlike most math education research, we report on the expansion of a year-long program, not on the results of short-term, single, assessments of achievement and anxiety. The strength of those assessments lies in the rigor of their statistical analysis. The weakness lies in their paucity of practical application. Importantly, they do not address the effects of current curricula on either the achievement or the anxiety they report. In contrast, the strength of our report lies precisely in application. We show that, in one school district, expansion of a quite different curricula can reduce anxiety by incrementing achievement.

There are two caveats. The first is that time constraints precluded pre- and post-testing of all classes. Since we visited each schools on a rotating basis, continuing professional development/demonstration/involvement in all the classrooms seemed more important to implementing the curriculum than testing all first graders.

The second caveat is that we could not compare classrooms with and without the new curriculum. It would have been unethical to leave any one school in the district without an intervention already proven (in the pilot school) to work so well.

However, an earlier study (Stokes, 2014b) can provide this comparison. First graders in the pilot group (using *Only the NUMERS Count*) outperformed the comparison group (using only *enVisionMATH*) on the (almost identical) post-test and on the district wide Renaissance STAR math test. On the latter, 71% of the pilot group scored above grade level. Only 30% of the comparison group did. That study did not include any math anxiety questions.

CONCLUSION: THE IMPORTANCE OF IMMERSION

Immersion – constant practice in manipulating numbers in order to understand them – was critical to the success of the current intervention. Success was measured in two ways: acquiring the math, not acquiring the anxiety. Immersed in the strictly mathematical, first graders in the four expansion schools – like those in the pilot (Stokes, 2014b) – had little difficulty with place-value, double-digit addition and subtraction, or composing, decomposing and recomposing addition combinations. Importantly, acquisition of the math did not include acquisition of anxiety. The results of both studies have three implications for early mathematical education.

- One, early immersion in the strictly mathematical can teach young students to think and problem solve like mathematicians, in numbers, symbols, and patterns.
- Two, early immersion can facilitate acquisition of procedural (how-to) knowledge and its product, conceptual (that/what) knowledge.
- Three, with early immersion, the math anxiety problem can be solved before it starts.[20]

[20] We have one other suggestion, based on conversations with teachers. Teachers should have a say in selecting a math curriculum. They are the ones who have to implement it. Most teachers would not select the curricula they are currently teaching.

REFERENCES

Adams, J.W., & Hitch, G.J. (1997). Working memory and children's mental addition. Journal of Experimental Child Psychology, 67, 21-38.

Anzai, Y., & Simon, H.A. (1979). The theory of learning by doing. Psychological Review, 86(2), 124-140).

Ashcraft, M.H., & Kirk, E.P. (2001). The relationships among working memory, math anxiety, and performance. Journal of Experimental Psychology: General, 130, 224-237.

Ashcraft, M.H., & Krause, J.A. (2007). Working memory, math performance, and math anxiety. Psychonomic Bulletin & Review, 14, 243-248.

Beilock, S.L, (2008). Math performance in stressful situation. Current Directions in Psychological Science, 17, 339-343.

Cardinale, J. (2019). Personal communication, June 6, 2019.

Chase, W.G., & Ericsson, K.A. (1982). Skill and working memory. In G.H. Bower (Ed.), The psychology of learning and motivation (Vol. 16, pp 1-58). NY: Academic Press.

Chase, W.G., & Simon, H.A. (1973). The mind's eye in chess. In W.G. Chase (Ed.), Visual information processing (pp. 215-281). NY: Academic Press.

Chi, M.T.H. (2011). Theoretical perspectives, methodological approaches, and trends in the study of expertise. In Y. Li (Ed.), Expertise in mathematics: An international perspective. (2nd Edition, pp.17-97). NY: Springer.

Chi, M.T.H., Glaser, R., & Farr, M.J. (1986). The nature of expertise. Hillsdale, NJ: Erlbaum.

DiLeo, L., & Stokes, P.D. (2019). Pilot to district: Rolling out an early math intervention. The New Jersey Mathematics Teacher, 77(1), 5-16.

Ericsson, K.A. (2006). The influence of experience and deliberate practice on the development of superior expert performance. In K.A. Ericsson, N. Charness, P.J. Feltovich, & Hoffman, R.R. (Eds.), The Cambridge handbook of expertise and expert performance (pp. 683-704). NY: Cambridge University Press.

Ericsson, K.A., & Kintsch, W. (1995). Long-term working memory. Psychological Review, 102, 211-245.

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place-value. Cognition and Instruction, 7, 343-403.

Harari, R.R., Vukovic, R.K., Bailey, S.P. (2013). Mathematics anxiety in young children: An exploratory study. Journal of Experimental Education, 81 (4), 538-555.

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. Journal for Research in Mathematics Education, 21, 33-46.

Johnson, K.J., & Swain, M. (Eds.) (1997). Immersion education: International perspectives. NY: Cambridge University Press.

Meece, J.L, Wigfield, A., & Eccles, J.S. (1990). Predictors of math anxiety and its influence on young adolescents' course enrollment intentions and performance in mathematics. Journal of Educational Psychology, 82 (1), 60-70.

National Research Council (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.

Newell, A., & Simon, H.A. (1972). Human problem solving. Englewood Cliffs, NJ: Prentice-Hall.

Nokes, T.J., Schunn, C.D., & Chi, M.T.H. (2010). Problem solving and human expertise. In International Encyclopedia of Education, Vol. 5 (pp. 265-272).

Papert, S. (1980). Mindstorms: Children, computers, and powerful ideas. NY: Basic Books.

Ramirez, G., Gunderson, E.A., Levine, S.C., & Beilock, S. (2013). Math anxiety, working memory, and math achievement in early elementary school. Journal of Cognition and Development, 14(2), 187-202.

Reitman, W. (1965). Cognition and thought. NY: Wiley.

Simon, H.A. (1973). The structure of ill-structured problems. Artificial Intelligence, 4, 181-201.

Simon, H.A. (1988). Learning from examples and by doing. Invited talk, presented October 3, 1988 at Teacher's College, Columbia University, NY.

Stokes, P.D. (2006). Creativity from constraints: The psychology of breakthrough. NY: Springer.

Stokes, P.D. (2014a). Using a creativity model to solve the place-value problem in kindergarten.

The International Journal of Creativity and Problem Solving, 24, 101-122.

Stokes, P.D. (2014b). How early is early enough? Solving the place-value problem in first grade.

The New Jersey Mathematics Teacher, 72, 30-40.

Stokes, P.D. (2016a). Thinking in patterns to solve multiplication, division, and fraction problems in second grade. Journal of Mathematics Education at Teacher's College, 7(2), 1-10.

Stokes, P.D. (2016b). Creativity from constraints in the performing arts. NY: CreateSpace.

Stokes, P.D., & Gibbert, M. (2019). Using paired constraints to solve the innovation problem. Switzerland: Springer Nature.

Suinn, R.M., Taylor, S., & Edwards, R.W. (1988). Suinn mathematics anxiety rating scale for elementary school students (MARS-E): Psychometric and normative data. Educational and Psychological Measurement, 48, 979-986.

Vukovic, R.K., Kieffer, M.J., Bailey, S.P., & Harari, R.R. (2013). Mathematics anxiety in young children: Concurrent and longitudinal associations with mathematical performance. Contemporary Educational Psychology, 38, 1-10.

Weisberg, R.W. (2006). Models of expertise in creative thinking: Evidence from case studies. In K.A. Ericsson, N. Charness, P.J. Feltovich, & R.R. Hoffman (Eds.), The Cambridge handbook of expertise and expert performance (pp. 761-788). NY: Cambridge University Press.

Young, C.B., Wu, S.S., & Menon, E. (2012). The neurological basis of math anxiety. Psychological Science, 23(5), 494-501.

Zhu, X., & Simon, H.A. (1987). Learning from examples and by doing. Cognition and Instruction, 4(3), 137-166.

APPENDIX

Questions for first grade teachers – June 2019

Name of teacher: ______________________

School: ____________________________

1. Do you enjoy teaching Only the NUMBERS Count?
2. If yes, what do you like about it? Please be specific.
3. What would you like changed or added to the program?
4. Do the children like leaning math with Only the NUMBERS Count?
5. If yes, what do they like about it? Again, please be specific.
6. Where are the children now with their math skills (e.g., single- or double-digit combinations, subtraction)?
7. Are you surprised at where they are (with learning math)?
8. Would you like Only the NUMBERS Count to be expanded as a complete math program? (e.g., with measurements, shapes, etc.)
9. What have you/have you told any other teachers about Only the NUMBERS Count?
10. Could you please ask your students the following questions, and indicate the numbers for each answer.

    a) How many of you like math? Number: _____________

    b) How many of you don't like math? Number: _________

ACKNOWLEDGEMENTS

Thanks to our Columbia University research assistants, Brian McIntyre and Emily Ringel, who did the observations and testing, to Lauren Launa, the curriculum coordinator who initiated the district-wide expansion, and especially to the principals, teachers, and children who made it both a success and a pleasure.